$$= \frac{\sin 72^\circ \cos 7^\circ}{\cos 18^\circ} = \frac{\cos 18^\circ}{\cos 18^\circ} \cdot \cos 7^\circ = \cos 7^\circ$$

$\cos 7^\circ = \cos 7^\circ$, т.е.

доказахме, че лявата страна е равна на дясната, с което задачата е решена.

Задача 9. Да се пресметне без четиризначна таблица $\sin 18^\circ$.

Решение: $2.18^\circ = 36^\circ$, $3.18^\circ = 54^\circ$, $36^\circ + 54^\circ = 90^\circ$.

Следователно $\sin 36^\circ = \sin(90^\circ - 54^\circ) = \cos 54^\circ$.

Полагаме $18^\circ = \alpha$; $\sin 36^\circ = \cos 54^\circ$, следователно $\sin 2\alpha = \cos 3\alpha$. Решаваме полученото уравнение

$$\sin 2\alpha - \cos 3\alpha = 0, \quad 2\sin\alpha\cos\alpha - (4\cos^3\alpha - 3\cos\alpha) = 0.$$

$$\cos\alpha(2\sin\alpha - 4\cos^2\alpha + 3) = 0, \qquad \cos\alpha \neq 0.$$

Следователно $2\sin\alpha - 4\cos^2\alpha + 3 = 0$.

$$2\sin\alpha - 4(1 - \sin^2\alpha) + 3 = 0$$

$2\sin\alpha - 4 + 4\sin^2\alpha + 3 = 0$. Полагаме $\sin\alpha = y$, където $y \in (0; 1)$.

$4y^2 + 2y - 1 = 0 \quad y_{1,2} = \dfrac{-1 \pm \sqrt{1+4}}{2} = \dfrac{-1 \pm \sqrt{5}}{4}$.

Решение ще бъде $\sin 18^\circ = \dfrac{-1 + \sqrt{5}}{4}$, тъй като $\sin 18^\circ$ е положително число.

Задача 10. Да се определи α без таблица, ако

$$\operatorname{tg}\alpha = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2.$$

Решение:

$$\operatorname{tg}\alpha = \sqrt{3}(\sqrt{2}+1) - \sqrt{2}(\sqrt{2}+1) = (\sqrt{3}-\sqrt{2})(\sqrt{2}+1) \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1} =$$

$$= \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1} = \frac{\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2} - \frac{1}{2}} = \frac{\sin 60^\circ - \sin 45^\circ}{\sin 45^\circ - \sin 30^\circ} =$$

$$= \frac{2\sin\frac{15^\circ}{2}\cos\frac{105^\circ}{2}}{2\sin\frac{15^\circ}{2}\cos\frac{75^\circ}{2}} = \frac{\sin\left(90^\circ - \frac{105^\circ}{2}\right)}{\cos\frac{75^\circ}{2}} = \frac{\sin\frac{75^\circ}{2}}{\cos\frac{75^\circ}{2}} = \operatorname{tg}\frac{75^\circ}{2}.$$

$\operatorname{tg}\alpha = \operatorname{tg}\dfrac{75^\circ}{2}$, следователно $\alpha = 37^\circ 30'$.

Задача 11. Да се докаже, че ако $\alpha + \beta + \gamma = 180^\circ$, то

а) $\sin\alpha + \sin\beta + \sin\gamma = 4\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}$;

б) $\cos\alpha + \cos\beta + \cos\gamma = 1 + 4\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}$;

в) $\operatorname{tg}\alpha + \operatorname{tg}\beta + \operatorname{tg}\gamma = \operatorname{tg}\alpha \operatorname{tg}\beta \operatorname{tg}\gamma$;

Решение:

а) $\sin\alpha + \sin\beta + \sin\gamma = \sin\alpha + \sin\beta + \sin[180^\circ - (\alpha+\beta)] =$

$$= \sin\alpha + \sin\beta + \sin(\alpha+\beta) = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} +$$

$$+2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2} = 2\sin\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2}\right) =$$

$$= 2\sin\frac{\alpha+\beta}{2}\cdot 2\cdot\cos\frac{\alpha}{2}\cos\frac{\beta}{2} = 4\sin\frac{180^\circ-\gamma}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2} =$$

$$= 4\sin\left(90^\circ - \frac{\gamma}{2}\right)\cos\frac{\alpha}{2}\cos\frac{\beta}{2} = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

б) $\cos\alpha + \cos\beta + \cos\gamma = 1 + 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

$\cos\alpha + \cos\beta + (\cos\gamma - 1) = \cos\alpha + \cos\beta - 2\sin^2\frac{\gamma}{2} =$

$= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} - 2\sin^2\frac{180^\circ-(\alpha+\beta)}{2} =$

$= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} - 2\sin^2[90 - \frac{(\alpha+\beta)}{2}] =$

$= 2\cos\frac{(\alpha+\beta)}{2}\cos\frac{\alpha-\beta}{2} - 2\cos^2\frac{\alpha+\beta}{2} =$

$= 2\cos\frac{\alpha+\beta}{2}\left[\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}\right] =$

$= 2\cos\frac{\alpha+\beta}{2}.2\sin\frac{\alpha}{2}\sin\frac{\beta}{2} = 4\cos\left[\frac{180^\circ-\gamma}{2}\right].\sin\frac{\alpha}{2}\sin\frac{\beta}{2} =$

$= 4\cos(90^\circ - \frac{\gamma}{2})\sin\frac{\alpha}{2}\sin\frac{\beta}{2} = 4\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$

т.е. $\cos\alpha + \cos\beta + \cos\gamma - 1 = 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

където $\cos\alpha + \cos\beta + \cos\gamma = 1 + 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

в) **I начин:** $\operatorname{tg}\alpha + \operatorname{tg}\beta + \operatorname{tg}\gamma = \operatorname{tg}\alpha + \operatorname{tg}\beta + \operatorname{tg}[180^\circ - (\alpha+\beta)] =$

$$= (\operatorname{tg}\alpha + \operatorname{tg}\beta) - \operatorname{tg}(\alpha+\beta) = \frac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{1 - \operatorname{tg}\alpha\operatorname{tg}\beta}\cdot(1 - \operatorname{tg}\alpha\operatorname{tg}\beta) -$$

$$-\operatorname{tg}(\alpha+\beta) = \operatorname{tg}(\alpha+\beta)(1 - \operatorname{tg}\alpha\operatorname{tg}\beta) - \operatorname{tg}(\alpha+\beta) =$$

$$= \operatorname{tg}(\alpha+\beta) - \operatorname{tg}\alpha\operatorname{tg}\beta\operatorname{tg}(\alpha+\beta) - \operatorname{tg}(\alpha+\beta) =$$

$$= -\operatorname{tg}\alpha\operatorname{tg}\beta\operatorname{tg}(180^\circ - \gamma) = \operatorname{tg}\alpha\operatorname{tg}\beta\operatorname{tg}\gamma;$$

II начин: $\operatorname{tg}\alpha+\operatorname{tg}\beta+\operatorname{tg}\gamma=\operatorname{tg}\alpha+\operatorname{tg}\beta+\operatorname{tg}\,[180^\circ-(\alpha+\beta)]=$

$$=\operatorname{tg}\alpha+\operatorname{tg}\beta-\operatorname{tg}(\alpha+\beta)=(\operatorname{tg}\alpha+\operatorname{tg}\beta)-\frac{\operatorname{tg}\alpha+\operatorname{tg}\beta}{1-\operatorname{tg}\alpha\operatorname{tg}\beta}=$$

$$=(\operatorname{tg}\alpha+\operatorname{tg}\beta)\left(1-\frac{1}{1-\operatorname{tg}\alpha\operatorname{tg}\beta}\right)=\frac{(\operatorname{tg}\alpha+\operatorname{tg}\beta).(1-\operatorname{tg}\alpha\operatorname{tg}\beta-1)}{1-\operatorname{tg}\alpha\operatorname{tg}\beta}=$$

$$=\operatorname{tg}(\alpha+\beta)(-\operatorname{tg}\alpha\operatorname{tg}\beta)=-\operatorname{tg}[180^\circ-\gamma]\operatorname{tg}\alpha\operatorname{tg}\beta=\operatorname{tg}\alpha\operatorname{tg}\beta\operatorname{tg}\gamma$$

Задача 12. Да се докаже, че ако $\alpha+\beta+\gamma=\frac{\pi}{2}$, то

$$\operatorname{cotg}\alpha+\operatorname{cotg}\beta+\operatorname{cotg}\gamma=\operatorname{cotg}\alpha\operatorname{cotg}\beta\operatorname{cotg}\gamma$$

Решение: $\operatorname{cotg}\alpha+\operatorname{cotg}\beta+\operatorname{cotg}\left[\frac{\pi}{2}-(\alpha+\beta)\right]=$

$$=\operatorname{cotg}\alpha+\operatorname{cotg}\beta+\operatorname{tg}(\alpha+\beta)=\frac{1}{\operatorname{tg}\alpha}+\frac{1}{\operatorname{tg}\beta}+\frac{\operatorname{tg}\alpha+\operatorname{tg}\beta}{1-\operatorname{tg}\alpha\operatorname{tg}\beta}=$$

$$=\frac{\operatorname{tg}\alpha+\operatorname{tg}\beta}{\operatorname{tg}\alpha.\operatorname{tg}\beta}+\frac{\operatorname{tg}\alpha+\operatorname{tg}\beta}{1-\operatorname{tg}\alpha\operatorname{tg}\beta}=(\operatorname{tg}\alpha+\operatorname{tg}\beta)\left(\frac{1}{\operatorname{tg}\alpha\operatorname{tg}\beta}+\frac{1}{1-\operatorname{tg}\alpha\operatorname{tg}\beta}\right)=$$

$$=(\operatorname{tg}\alpha+\operatorname{tg}\beta)\frac{(1-\operatorname{tg}\alpha\operatorname{tg}\beta+\operatorname{tg}\alpha\operatorname{tg}\beta)}{\operatorname{tg}\alpha\cdot\operatorname{tg}\beta(1-\operatorname{tg}\alpha\operatorname{tg}\beta)}=\frac{\operatorname{tg}\alpha+\operatorname{tg}\beta}{1-\operatorname{tg}\alpha\operatorname{tg}\beta}.\frac{1}{\operatorname{tg}\alpha\operatorname{tg}\beta}=$$

$$=\operatorname{tg}(\alpha+\beta).\operatorname{cotg}\alpha\operatorname{cotg}\beta=\operatorname{tg}\left[\frac{\pi}{2}-\gamma\right]\operatorname{cotg}\alpha\operatorname{cotg}\beta=\operatorname{cotg}\alpha\operatorname{cotg}\beta\operatorname{cotg}\gamma$$

Задача 13. Ако $\sin\alpha+\cos\alpha=a$, да се пресметне

а) $\sin\alpha.\cos\alpha$, б) $\sin^3\alpha+\cos^3\alpha$, в) $\operatorname{tg}\alpha+\operatorname{cotg}\alpha$,
г) $\sin^4\alpha+\cos^4\alpha$, д) $\sin^6\alpha+\cos^6\alpha$.

Решение: а) $\sin\alpha+\cos\alpha=a,\quad(\sin\alpha+\cos\alpha)^2=a^2$

$$\sin^2\alpha+\cos^2\alpha+2\sin\alpha\cos\alpha=a^2$$

$$1+2\sin\alpha\cos\alpha=a^2,\quad \sin\alpha\cos\alpha=\frac{a^2-1}{2};$$

б) $\sin^3\alpha+\cos^3\alpha=(\sin\alpha+\cos\alpha)(\sin^2\alpha-\sin\cos\alpha+\cos^2\alpha)=$

$$=a\left(1-\frac{a^2-1}{2}\right);$$

в) $\operatorname{tg}\alpha+\operatorname{cotg}\alpha=\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}=\frac{\sin^2\alpha+\cos^2\alpha}{\sin\alpha\cos\alpha}=\frac{1}{\sin\alpha\cos\alpha}=$

$$=\frac{2}{a^2-1};$$

г) $\sin^4\alpha+\cos^4\alpha=(\sin^2\alpha)^2+(\cos^2\alpha)^2=(\sin^2\alpha+\cos^2\alpha)^2-$

$$-2\sin^2\alpha.\cos^2\alpha=1-2\sin^2\alpha.\cos^2\alpha=1-2(\sin\alpha\cos\alpha)^2=$$

$$=1-2(\frac{a^2-1}{2})^2=1-\frac{2(a^2-1)^2}{4}=1-\frac{(a^2-1)^2}{2};$$

д) $\sin^6\alpha + \cos^6\alpha = (\sin^2\alpha)^3 + (\cos^2\alpha)^3 = (\sin^2\alpha + \cos^2\alpha)^3 -$

$-3\sin^2\alpha\cos^2\alpha(\sin^2\alpha + \cos^2\alpha) = 1 - 3\sin^2\alpha\cos^2\alpha = 1 - 3\left(\dfrac{a^2-1}{2}\right)^2.$

Задача 14. Да се изключи параметъра α от двойката равенства :

а) $\left|\begin{array}{l} \operatorname{tg}\alpha + \operatorname{cotg}\alpha = a \\ \operatorname{tg}^2\alpha + \operatorname{cotg}^2\alpha = b \end{array}\right.$ б) $\left|\begin{array}{l} \sin\alpha + \cos\alpha = a \\ \sin^3\alpha + \cos^3\alpha = b \end{array}\right.$

Решение:

а) $\left|\begin{array}{l} \dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\cos\alpha}{\sin\alpha} = a \\ \dfrac{\sin^2\alpha}{\cos^2\alpha} + \dfrac{\cos^2\alpha}{\sin^2\alpha} = b \end{array}\right.$ $\left|\begin{array}{l} \dfrac{\sin^2\alpha + \cos^2\alpha}{\sin\alpha\cos\alpha} = a \\ \dfrac{\sin^4\alpha + \cos^4\alpha}{\sin^2\alpha\cos^2\alpha} = b \end{array}\right.$

$$\left|\begin{array}{l} \dfrac{1}{\sin\alpha\cos\alpha} = a \qquad \to \sin\alpha\cos\alpha = \dfrac{1}{a} \\ \dfrac{1 - 2\sin^2\alpha\cos^2\alpha}{\sin^2\alpha\cos^2\alpha} = b \end{array}\right.$$

$\dfrac{1 - 2\cdot\dfrac{1}{a^2}}{\left(\dfrac{1}{a}\right)^2} = b$ или $\dfrac{a^2-2}{a^2\left(\dfrac{1}{a}\right)^2} = b$. Следователно $a^2 - 2 = b$.

б) $\left|\begin{array}{l} \sin\alpha + \cos\alpha = a \\ \sin^3\alpha + \cos^3\alpha = b \end{array}\right.$

$\sin^3\alpha + \cos^3\alpha = (\sin\alpha + \cos\alpha)^3 - 3\sin\alpha\cos\alpha(\sin\alpha + \cos\alpha)$

$(\sin\alpha + \cos\alpha)^3 - 3\sin\alpha\cos\alpha(\sin\alpha + \cos\alpha) = b$. Полагаме

$$\sin\alpha + \cos\alpha = a.$$

$$a^3 - 3\sin\alpha\cos\alpha\, a = b \tag{1}$$

От $\sin\alpha + \cos\alpha = a$, $(\sin\alpha + \cos\alpha)^2 = a^2$.

$$\sin^2\alpha + \cos^2\alpha + 2\sin\alpha\cos\alpha = a^2$$

$\sin\alpha\cos\alpha = \dfrac{a^2-1}{2}$, заместваме в (1) и получаваме

$$a^3 - 3\frac{a^2-1}{2}\cdot a = b.$$

Задача 15. Да се елиминира α от системата:

$$\left| \begin{array}{l} \sin\alpha + \dfrac{1}{\cos\beta} = a \\ \cos\alpha + \dfrac{1}{\sin\beta} = b \end{array} \right.$$

Решение: Определяме $\sin\alpha$ от първото равенство, $\cos\alpha$ – от второто и заместваме в $\sin^2\alpha + \cos^2\alpha = 1$.

$\sin\alpha = a - \dfrac{1}{\cos\beta}, \quad \cos\alpha = b - \dfrac{1}{\sin\beta}, \quad \sin^2\alpha + \cos^2\alpha = 1.$

Получаваме $\left(a - \dfrac{1}{\cos\beta}\right)^2 + \left(b - \dfrac{1}{\sin\beta}\right)^2 = 1.$

Задача 16. Да се определи вида на триъгълник ABC, ако за ъглите му α и β са дадени следните зависимости:

$$\left| \begin{array}{l} \sin\alpha + \sin\beta = \sqrt{3} \\ \cos\alpha + \cos\beta = 1. \end{array} \right.$$

Решение: Повдигаме на втора степен двете уравнения и получаваме

$$+\left| \begin{array}{l} \sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta = 3 \\ \cos^2\alpha + 2\cos\alpha\cos\beta + \cos^2\beta = 1 \end{array} \right.$$

$1 + 2(\sin\alpha\sin\beta + \cos\alpha\cos\beta) + 1 = 4, \quad 2(\sin\alpha\sin\beta + \cos\alpha\cos\beta) = 2.$

$\sin\alpha\sin\beta + \cos\alpha\cos\beta = 1, \quad \cos(\alpha - \beta) = 1 \quad \to \alpha - \beta = 0.$

Следователно $\alpha = \beta$, от което следва, че $\triangle ABC$ е равнобедрен.

Задача 17. Докажете, че ако $3\sin\beta = \sin(2\alpha + \beta)$, то $\operatorname{tg}(\alpha + \beta) = 2\operatorname{tg}\alpha$.

Решение: $3\sin\beta = \sin 2\alpha\cos\beta + \cos 2\alpha\sin\beta.$

$$3\sin\beta = 2\sin\alpha\cos\alpha\cos\beta + (1 - 2\sin^2\alpha)\sin\beta$$
$$3\sin\beta = 2\sin\alpha\cos\alpha\cos\beta + \sin\beta - 2\sin^2\alpha\sin\beta$$
$$3\sin\beta - \sin\beta + 2\sin^2\alpha\sin\beta = 2\sin\alpha\cos\alpha\cos\beta$$
$$2\sin\beta + 2\sin^2\alpha\sin\beta = 2\sin\alpha\cos\alpha\cos\beta$$
$$2\sin\beta(1 + \sin^2\alpha) = 2\sin\alpha\cos\alpha\cos\beta \quad (:\ \cos\beta,\ \beta \neq (2k+1)\frac{\pi}{2}\ )$$