$\sqrt{1-b^2}\,(1+3b)=3(b+1)\left(b-\frac{1}{3}\right);$

$(1-b)(1+b)(1+3b^2)=9(b+1)^2\left(b-\frac{1}{3}\right)^2 \Big|:(1+b)\neq 0;$

$(1-b)(1+3b)^2=(b+1)(3b-1)^2;\ (1-b)(1+6b+9b^2)=(b+1)(9b^2-6b+1);$

$1+6b+9b^2-b-6b^2-9b^3=9b^3-6b^2+b+9b^2-6b+1;$

$18b^3-10b=0;\ 2b(9b^2-5)=0;\ b=0$ - не е решение на задачата.

$9b^2=5;\ b^2=\frac{5}{9};\ b=\frac{\sqrt{5}}{3};$

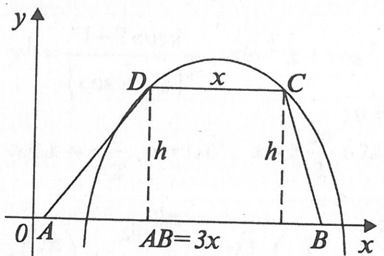
$$MN=\sqrt{\frac{1}{2}\left(\sqrt{1-\frac{5}{9}}+\frac{\sqrt{5}}{3}-1\right)^2\left(1+\frac{\sqrt{5}}{3}\right)}=\sqrt{\frac{1}{2}\left(\sqrt{\frac{4}{9}}+\frac{\sqrt{5}}{3}-1\right)^2\left(\frac{3+\sqrt{5}}{3}\right)}=$$

$$=\sqrt{\frac{1}{2}\left(\frac{2}{3}+\frac{\sqrt{5}}{3}-1\right)^2\left(\frac{3+\sqrt{5}}{3}\right)}=\sqrt{\frac{1}{2}\left(\frac{\sqrt{5}-1}{3}\right)^2\left(\frac{3+\sqrt{5}}{3}\right)}=\frac{\sqrt{5}-1}{3}\sqrt{\frac{3+\sqrt{5}}{6}};$$

$MN=\frac{\sqrt{5}-1}{3\sqrt{6}}\sqrt{3+\sqrt{5}};\ \frac{\sqrt{5}-1}{3\sqrt{6}}\sqrt{3+\sqrt{5}}\le\frac{2\sqrt{3}}{9};\ \frac{5-2\sqrt{5}+1}{9.6}.(3+\sqrt{5})\le\frac{4.3}{81};$

$\frac{(6-2\sqrt{5})(3+\sqrt{5})}{36}\le\frac{4}{27};\frac{(3-\sqrt{5})(3+\sqrt{5})}{27}\le\frac{4}{27};9-5\le 4$, т.е. $MN\le\frac{2\sqrt{3}}{9}$.

Стр.77, Зад.4. $y=-x^2+78x-7$; Означаваме $ABCD$ - трапец, където $AB=3CD$. $S_{ABCD}=?-\max$.



Нека т.D има координати $D(a;f(a))$, където $f(a)=-a^2+8a-7$, което ще бъде височината h на трапеца. $CD=x$, $AB=3x$.

Тогава координатите на точка C са $C(a+x;f(a+x))$, където $f(a+x)=-(a+x)^2+8(a+x)-7=h$, защото точките C и D имат равни ординати или $f(a)=f(a+x)$.

$-a^2+8a-7=-(a+x)^2+8(a+x)-7;$

$-a^2+8a-7=-(a^2+2ax+x^2)+8a+8x-7;$

$-a^2+8a-7=-a^2-2ax-x^2+8a+8x-7;\ -2ax-x^2+8x=0;$

$x^2+2ax-8x=0;\ 2ax=8x-x^2;\ a=\frac{8x-x^2}{2x}=4-\frac{x}{2};$

$$S=\frac{x+3x}{2}h=2xh=2x(-a^2+8a-7)=2x\left[-4\left(4-\frac{x^2}{2}\right)^2+8\left(4-\frac{x}{2}\right)-7\right]=$$

$$=2x\left[-\left(16-4x+\frac{x^2}{4}\right)+32-4x-7\right]=2x\left(-16+4x-\frac{x^2}{2}+32-4x-7\right)=$$

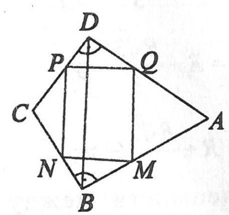
$=2x\left(-\frac{x^2}{4}+9\right)=-\frac{x^3}{2}+18x;\quad S^{|}=-\frac{3x^2}{2}+18=0;\quad S^{||}=-3x<0\ \max;$

$-\frac{3x^2}{2}+18=0;\quad 3x^2=36;\quad x^2=12;\quad x=\sqrt{12};\quad S=-\frac{x^3}{2}+18x;$

$S_{\max}=-\frac{(\sqrt{12})^3}{2}+18\sqrt{12}=-\frac{12\sqrt{12}}{2}+18\sqrt{12}=12\sqrt{12}$.

Стр.77, Зад.5. От ΔADC, където $2^2+1^2=(\sqrt{5})^2$ следва, че $\measuredangle CDA=\measuredangle CBA=90^0$; $ABCD$ - делтоид; $BD\perp AC$. Тогава $MNPQ$ - е правоъгълник със страни съответно успоредни на диагоналите AC и BD.

$AM=kAB$; $CN=xBC$; $k=\frac{AM}{AB}$; $x=\frac{CN}{BC}$; $k\in(0;1)$; $x\in(0;1)$. От $AD=AB=2\to AM=k.2=2k$. $\Rightarrow MB=QD=2-2k=2(1-k)$;



От $CD=BC=1$ следва $CN=x.BC=x.1=PC$;

$NB=DP=CD-PC=1-x$;

$S_{MNPQ}=S(x)=S_{ABCD}-2S_{MNB}-S_{AMQ}-S_{NCP}=$

$=2-2.\frac{BN.MB}{2}-\frac{NC.PC.\sin\angle BCD}{2}-\frac{AM.AQ\sin\angle BAD}{2}=$

$=2-\frac{2(1-x).2(1-k)}{2}-\frac{x.x\sin(\pi-\alpha)}{2}-\frac{2k.2k.\sin\alpha}{2}=$

$$2-2(1-x-k+kx)-\frac{x^2}{2}\sin\alpha-2k^2\sin\alpha=2x+2k-2kx-\sin\alpha\left(\frac{x^2}{2}+2k^2\right)=$$

$$=-\frac{2}{5}x^2-\frac{8}{5}k^2-2x(k-1)+2k=-\frac{2}{5}x^2-2x(k-1)+2k-\frac{8}{5}k^2.$$

$$S(x)=-\frac{2}{5}x^2-2x(k-1)+2k-\frac{8}{5}k^2=\left(-2x^2-10x(k-1)+10k-8k^2\right)\frac{1}{5}=$$

$$-\frac{1}{5}.\left[2x^2+10x(k-1)+8k^2-10k\right]=-\frac{2}{5}\left[x^2+5x(k-1)+4k^2-5k\right]=$$

$$=-\frac{2}{5}\left[x^2+\frac{2.5(k-1)}{2}.x+\left(\frac{5(k-1)}{2}\right)^2-\left(\frac{5(k-1)}{2}\right)^2-4k^2-5k\right]=$$

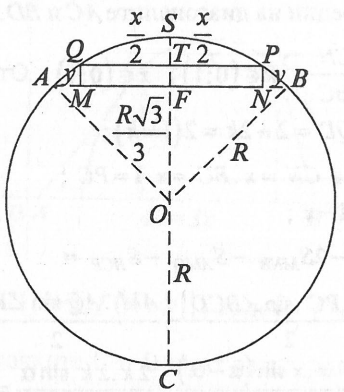
$$=-\frac{2}{5}\left[\left(x+\frac{5}{2}(k-1)\right)^2+4k^2-5k-\frac{25}{4}(k-1)^2\right]$$

Най-голяма стойност има $f(x)$ за $x+\frac{5}{2}(k-1)=0$ или $x=\frac{5}{2}(1-k)$ и

най-голяма стойност е

$$-\frac{2}{5}\left(4k^2-5k-\frac{25}{4}k^2+\frac{25}{2}k-\frac{25}{4}\right)=-\frac{2}{5}\left(\frac{-9k^2}{4}+\frac{15}{2}k-\frac{25}{4}\right)=$$

$$=\frac{2}{5}\left[\left(\frac{3k}{2}\right)^2-2.\frac{5}{4}k+\left(\frac{5}{2}\right)^2\right]=\frac{2}{5}\left(\frac{3}{2}k-\frac{5}{2}\right)^2=\frac{2}{4.5}(3k-5)^2=\frac{1}{10}(3k-5)^2$$



Стр.77, Зад.6.

$MNPQ$ - правоъгълник. $PT=QT=\frac{x}{2}$;

$NP=MQ=TF=y$; $OF=\frac{R\sqrt{3}}{3}$.

$ST=OS-(OF+TF)=R-\frac{R\sqrt{3}}{3}-y$;

$CT=OC+OF+TF=R+\frac{R\sqrt{3}}{2}+y$;

От метрични зависимости между отсечки в окръжност $PT.QT=CT.ST$ или

$$\frac{x}{2}.\frac{x}{2}=\left(y+\frac{R\sqrt{3}}{3}+R\right)\left(R-y-\frac{R\sqrt{3}}{3}\right);\ \frac{x^2}{4}=\left(y+\frac{3R+R\sqrt{3}}{3}\right)\left(\frac{3R-R\sqrt{3}}{3}-y\right);$$

$$\frac{x^2}{4}=\left(3y+3R+R\sqrt{3}\right)\left(3R-R\sqrt{3}-3y\right).\frac{1}{9};$$

$$\frac{9x^2}{4}=9yR+9R^2+3R^2\sqrt{3}-3yR\sqrt{3}-3R^2\sqrt{3}-R^2-9y^2-9yR-3yR\sqrt{3};$$

$$\frac{9x^2}{4}=6R^2-6Ry\sqrt{3}-9y^2\ |:3;\ \frac{3x^2}{4}=2R^2-2Ry\sqrt{3}-3y^2;$$

$$x^2=\frac{4}{3}\left(2R^2-2Ry\sqrt{3}-3y^2\right);\ x=\frac{2}{\sqrt{3}}\sqrt{2R^2-2Ry\sqrt{3}-3y^2};$$

$$S=xy=\frac{2}{3}\sqrt{\underbrace{2R^2y^2-2Ry^3\sqrt{3}-3y^4}_{\varphi}};$$

$$\varphi'=\left(2R^2y^2-2Ry^3\sqrt{3}-3y^4\right)'=4R^2y-6Ry^2\sqrt{3}-12y^3;$$

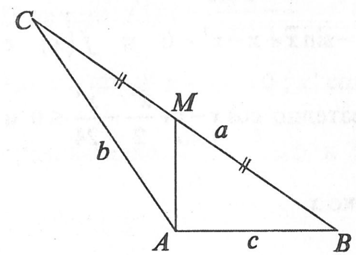
$$4R^2y-6Ry^2\sqrt{3}-12y^3=0\ |:y\neq 0;\ 4R^2-6Ry\sqrt{2}-12y^2=0;$$

$$6y^2+3Ry\sqrt{3}-2R^2=0;\ y_{1,2}=\frac{-3R\sqrt{3}\pm\sqrt{27R^2+48R^2}}{12}=\frac{-3R\sqrt{3}\pm 5\sqrt{3}R}{12};$$

$$y=\frac{2R\sqrt{3}}{12}=\frac{R\sqrt{3}}{6};\ x=\frac{2}{\sqrt{3}}\sqrt{2R^2-2R\sqrt{3}.\frac{R\sqrt{3}}{6}-\frac{3.3R^2}{36}}=$$

$$=\frac{2}{\sqrt{3}}\sqrt{2R^2-R^2-\frac{R^2}{4}}=\frac{2}{\sqrt{3}}\sqrt{\frac{3}{4}R^2};\ x=\frac{2}{\sqrt{3}}.\frac{\sqrt{3}}{2}.R=R.$$

Стр.77, Зад.7. При $BC=a=1$, $b=\sqrt{1-3c^2}$. $P_{ABC}=2p=1+c+\sqrt{1-3c^2}$;



$f(c)=c+\sqrt{1-3c^2}$.

От ΔAMB $AB<BM$, т.е. $0<c<\frac{1}{2}$.

$f'(c)=1-\frac{3c}{\sqrt{1-3c^2}}=0$; $\sqrt{1-3c^2}=3c$;

$1-3c^2=9c^2$; $12c^2=1$; $c=\pm\frac{1}{\sqrt{12}}$;